

Oxidation of $D = 3$ cosets and Bonnor dualities in $D \leq 6$

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Bonnor's map in General Relativity is duality between (dimensionally reduced) vacuum gravity and static truncation of electro-vacuum theory. It was used as a tool to generate an exact solution of electro-vacuum from some vacuum solution. It can be expected that similar dualities will be useful for solution generation in higher-dimensional theories too. Here we study such maps within a class of theories in dimensions $D \leq 6$ using oxidation of $D = 3$ cosets and consistent truncation of the corresponding theories. Our class includes those theories whose $D = 3$ symmetries are subgroups of $G = O(5, 4)$. It contains six-dimensional minimal supergravity, five-dimensional minimal and $U(1)^3$ supergravities and a number of four-dimensional theories which attracted attention recently in the search of exact solutions. We give explicit reduction/truncation formulas relating different theories in dimensions $D = 4, 5, 6$ in terms of metrics and matter fields and discuss various alternative duality chains between them.

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I. INTRODUCTION

Toroidal dimensional reduction of bosonic sectors of supergravities in $D \leq 11$ dimensions to $D = 3$ leads to sigma models on coset spaces [1–3]. The D -dimensional theories generically contain the metric and the scalar (with no potentials), vector and higher-rank form fields. Assuming the existence of $D - 3$ commuting Killing vectors one reduces them to three-dimensional theories with scalar and vector fields only, the latter

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then being replaced by additional scalars using three-dimensional Hodge dualisation. The final set of scalar fields (in General Relativity usually called potentials) contains the same number of degrees of freedom as the initial theory in D dimensions restricted to all fields depending only on $D - 3$ coordinates. Due to dualisation involved in this derivation the map between the initial and the three-dimensional scalar variables is not point-like, however. These scalars in many cases parameterize homogeneous spaces G/H where G is some Lie group (semi-simple in most interesting cases) and H is the isotropy subgroup. A rather complete list of such cosets was given by Breitenlohner, Maison and Gibbons [4], for more detailed discussion see [5]).

The isometry groups G of these cosets (often called hidden symmetries) are extremely useful in the search of exact solutions in gravity/supergravity. Historically, the use of $D = 3$ sigma models for solution generation in four dimensions goes back to the papers of Ehlers [6] (vacuum Einstein equations), Neugebauer and Kramer [7] (electrovacuum), Maison [8] and Clément [9–11] (Kaluza-Klein theory). During the past two decades similar methods were developed for various supergravity/string effective actions in four [12–20] and five dimensions [21–26]. Special attention was paid to the description of BPS states as nilpotent orbits of coset isometries [10, 11, 27–30]. A particular interest in $D = 5$ theories was motivated by the discovery of black rings and related objects. The conjectured existence of similar solutions in $D = 6$ stimulates further investigation of $D = 6$ theories along the same lines.

Apart from using symmetries of a *given theory*, generating techniques can be extended using maps between *different theories* described by the same cosets, or their consistent truncations. A simple example is provided by Bonnor’s map [31] in General Relativity, relating the static subspace of $D = 4$ Einstein-Maxwell (EM) theory and vacuum Einstein gravity, both having the same hidden symmetry $SL(2, R)$. By mapping the target space potentials related to these two theories, Bonnor was able to derive an exact static magnetic dipole solution of EM theory. Similar methods were applied to Kaluza-Klein theory [9] and to Einstein-Maxwell-dilaton theory with arbitrary dilaton coupling [32].

A convenient framework to reveal Bonnor-type maps in more general cases and higher dimensions is provided by the theory of oxidation — determination of higher-dimensional theories whose toroidal compactification to three dimensions leads to a given coset. The basic features of oxidation were described by Cremmer, Julia, Lu and Pope [33], an extensive discussion from the group-theoretical viewpoint was later given by Keurentjes [34, 35]. Generically a coset can be oxidized to various higher dimensions up to an ultimate oxidation endpoint which can be identified using some rules. One such rule follows from the fact that D -dimensional vacuum Einstein gravity leads to the coset $SL(D - 2, R)/O(D - 2)$ [3, 8]. Conversely, a maximal oxidation of the $SL(n, R)/O(n)$ coset is $(D = n + 2)$ - dimensional General Relativity. As was observed in [33], a similar rule holds for the maximally non-compact forms of D_n ($G = O(n, n)$), and B_n

($G = O(n+1, n)$), with an exception for B_3 , whose standard oxidation endpoint is $D = 5$, but which can be further oxidized to $D = 6$ leading to minimal $D = 6$ supergravity with a self-dual three-form. It was also argued that the C_n sequence ($G = Sp(2n, R)$) as well as some of the non-maximally non-compact forms of $G = O(p, q)$ and $G = SU(p, q)$ cannot be oxidized beyond four dimensions. It can be expected that lower-dimensional theories in the oxidation chain of a given coset are related by toroidal reduction of the theory corresponding to the oxidation endpoint highest-dimensional member. In many cases this reduction is accompanied by some non-trivial rearrangements of the matter fields.

The list of dualities is substantially enhanced if we include oxidations of *invariant subspaces* of a given coset, and this is our basic idea. Actually, in the original Bonnor's case it was duality between $D = 4$ vacuum gravity and the *static truncation* of $D = 4$ Einstein-Maxwell theory. Theories corresponding to different subspaces are related to the parent theory by consistent truncations. Therefore at each oxidation level we obtain a chain of truncations which in turn can be reduced to an extended set of lower dimensional theories. Mappings between different members of the whole set of theories constitute the generalized Bonnor-type dualities, which we investigate in this paper. In contrast with the original Bonnor map, they describe relations between theories in different space-time dimensions.

II. THE SETUP

Our goal is to systematically study the mappings between different theories obtained by oxidation of $D = 3$ cosets which can be embedded into $O(5, 4)/(O(5) \times O(4))$ whose oxidation endpoint is $D = 6$. It is worth noting that this is not the largest coset oxidizing up to $D = 6$: the coset of the exceptional group F_4 , also having the oxidation endpoint $D = 6$, is larger. Rather, our choice is motivated by the simplicity (and at the same time multiplicity) of the resulting duality chains. More specifically, our list of theories includes the following:

- Heterotic string effective actions [36] restricted to p vector fields, alternatively called Einstein-Maxwell-Dilaton-Axion (EM_pDA) theories

$$\mathcal{L} = \sqrt{g} \left(R - \frac{1}{2}(\partial\psi)^2 - \frac{1}{4}e^{\beta\psi}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{12}e^{2\beta\psi}G_{\mu\nu\rho}\hat{G}^{\mu\nu\rho} \right). \quad (2.1)$$

with

$$F_{\mu\nu}^a = 2A_{[\nu, \mu]}^a, \quad G_{\mu\nu\rho} = 3(B_{[\mu\nu\rho]} + \frac{1}{2}F_{[\mu\nu}^a A_{\rho]}^a), \quad (2.2)$$

where sum over a from one to p is understood in all terms quadratic in the vector fields, and the constant β depends on dimension. Our set will include the cases $p = 0$ (no vector fields, EDA), $p = 1$, ($EMDA$), $p = 2$ (EM_2DA) and $p = 3$ (EM_3DA).

- Minimal supergravities (*MSG*), whose bosonic lagrangians are

$$\mathcal{L}_6 = \sqrt{g_6} \left(R - \frac{1}{12} G_{\mu\nu\lambda} G^{\mu\nu\lambda} \right), \quad (2.3)$$

in six dimensions,

$$\mathcal{L}_5 = \sqrt{g_5} \left(R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) - \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A_\lambda, \quad (2.4)$$

in five, and the pure Einstein term in four dimensions.

- $U(1)^3$ -supergravity in $D = 5$:

$$\mathcal{L} = \sqrt{g_5} \left(R - \frac{1}{2} \mathcal{G}_{IJ} (\partial X^I) (\partial X^J) - \frac{1}{4} \mathcal{G}_{IJ} F_{\mu\nu}^I F_{\mu\nu}^J \right) - \frac{1}{24} \epsilon^{\mu\nu\lambda\rho\tau} \delta_{IJK} F_{\mu\nu}^I F_{\lambda\rho}^J A_\tau^K, \quad (2.5)$$

where the three scalars $X_I, I = 1, 2, 3$, are constrained by $X_1 X_2 X_3 = 1$, $F^I = dA^I$, and $\mathcal{G}_{IJ} = \text{diag}((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2})$, $\delta_{IJK} = 1$ for the permutation of the indices 1, 2, 3 and zero otherwise.

- Einstein-Maxwell (*EM*) theory in $D = 4$.
- Vacuum Einstein gravity (*VG*),

Their hidden symmetry groups G are listed in the Table

	D=6	D=5	D=4
<i>VG</i>	$SL(4, R)$	$SL(3, R)$	$SL(2, R)$
<i>EDA</i>	$O(4, 4)$	$O(3, 3)$	$O(2, 2)$
<i>EMDA</i>	$O(5, 4)$	$O(4, 3)$	$O(3, 2) \sim Sp(4, R)$
<i>EM₂DA</i>	—	$O(5, 3)$	$O(4, 2) \sim SU(2, 2)$
<i>EM₃DA</i>	—	—	$O(5, 2)$
<i>MSG</i>	$O(4, 3)$	$G_{2,(2)}$	$SL(2, R)$
$U(1)^3$ <i>SG</i>	—	$O(4, 4)$	—
<i>EM</i>	—	—	$SU(2, 1)$

(2.6)

Empty spaces for *EM* theories correspond to the absence of non-trivial hidden symmetries. Empty spaces for the $EM_p DA$ series for $D = 5, 6$ correspond to G beyond the ultimate group $O(5, 4)$, namely, $EM_2 DA$ in $D = 6$ has $G = O(6, 4)$, while $EM_3 DA$ has $G = O(7, 4)$ in $D = 6$ and $G = O(6, 3)$ in $D = 5$.

These theories are related by duality mappings of two sorts. The first refers to different oxidation points of the same coset. Such theories are related by toroidal dimensional reduction from the oxidation endpoint.

The second refers to theories whose hidden symmetries are subgroups of the largest group. These are connected by consistent truncations of the largest theory. As a rule for consistent truncations we use a slightly modified version of that given in [4]:

Proposition. Let $\mathcal{G} = G/H$ where G is a simple group and H is its maximal compact subgroup¹. Then if $G_1 \subset G$ is a subgroup, and $H_1 \subset H$ is the maximal compact subgroup of G_1 , one has

- i) $\mathcal{G}_1 = G_1/H_1$ is a totally geodesic subspace of \mathcal{G} ;
- ii) the D -dimensional oxidation of \mathcal{G} can be consistently truncated to \mathcal{G}_1 .

Combining toroidal dimensional reduction and consistent truncations, one is able to derive all the listed theories from the six-dimensional (one vector) heterotic effective action (*EMDA*) with hidden symmetry $G = O(5,4)$. This theory admits the sequence of consistent truncations first to (*EDA*) with $G = O(4,4)$, then to minimal six-dimensional supergravity (*MSG*) with $G = O(4,3)$ and finally to vacuum gravity with $G = SL(4,R)$. We then perform dimensional reduction of this set of theories to five dimensions. The “parent” theory $G = O(5,4)$ compactified on S^1 goes beyond our list (containing three vectors, a two-form and three scalars), but it can be consistently truncated to $U(1)^3$ – five-dimensional supergravity with $G = O(4,4)$ and to EM_2DA with $G = O(5,3)$, giving rise to two further chains of truncations. One chain goes through $U(1)^3$ *SG* which further truncates to *EMDA* with $G = O(4,3)$, and then to either to *EDA* with $G = SL(4,R) \sim O(3,3)$, or to minimal supergravity with $G = G_{(2,2)}$, which can be further truncated only to vacuum gravity with $G = SL(3,R)$. The other chain goes through EM_2DA , meeting with the first one at the *EMDA* level. Compactification of these theories on S^1 and their consistent truncations in four dimensions complete our list (2.6).

Many of the above relations between the mentioned theories were discussed earlier, most notably in [33, 37]. Our goal here is to give explicit realizations in terms of original field variables, aiming to open new possibilities for solution generation technique.

III. TRUNCATIONS IN SIX DIMENSIONS

We start with the largest $D = 3$ coset $O(5,4)/(O(5) \times O(4))$ admitting $D = 6$ as oxidation endpoint. It is a 20-dimensional homogeneous space. The corresponding six-dimensional theory is the heterotic string effective action with one vector field. The set of fields contain the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, the three-form field $\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ (axion), the Maxwell two-form $\hat{F}_{\hat{\mu}\hat{\nu}}$ and the dilaton $\hat{\psi}$, so we call this theory *EMDA* (the full heterotic string

¹ For definiteness we consider compactification on a purely space-like torus. Reducing on the time direction leads to non-compact forms of H without changing the desired maps. We will also write the space-time measure as $\sqrt{\bar{g}}d^Dx$, including a minus sign into the definition of g .

$U(1)^{16}$ effective action toroidally reduced to six dimensions contains 24 vectors). The Lagrangian reads

$$\mathcal{L}_{EMDA}^6 = \sqrt{\hat{g}} \left(\hat{R} - \frac{1}{2}(\partial\hat{\psi})^2 - \frac{1}{4}e^{2\beta\hat{\psi}}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}} - \frac{1}{12}e^{2\beta\hat{\psi}}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{G}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right). \quad (3.1)$$

($\beta^2 = 1/2$) with

$$\hat{F}_{\hat{\mu}\hat{\nu}} = 2\hat{A}_{[\hat{\nu},\hat{\mu}]}, \quad \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3(\hat{B}_{[\hat{\mu}\hat{\nu},\hat{\rho}]} + \frac{1}{2}\hat{F}_{[\hat{\mu}\hat{\nu}}\hat{A}_{\hat{\rho}]}). \quad (3.2)$$

This theory admits consistent truncations according to the chain of subgroups

$$O(5,4) \longrightarrow O(4,4) \longrightarrow O(4,3) \longrightarrow O(3,3) \sim SL(4,R), \quad (3.3)$$

labeling the sequence of invariant subspaces of the initial coset:

- $O(4,4)/(O(4) \times O(4))$ (dimension 16),
- $O(4,3)/(O(4) \times O(3))$ (dimension 12),
- $SL(4,R)/(O(3) \times O(3))$ (dimension 9).

All of them admit $D = 6$ as the oxidation endpoint (the case $O(4,3)/(O(4) \times O(3))$ being exceptional [33]). By dimensionality, it is clear that the first coset correspond to setting zero the Maxwell field $\hat{F}_{\hat{\mu}\hat{\nu}} = 0$ which under reduction to $D = 3$ generates four scalars. It is a consistent truncation of the Lagrangian (3.1), as can be easily seen from the equations of motion. The truncated theory is *EDA*:

$$\mathcal{L}_{EDA}^6 = \sqrt{\hat{g}} \left(\hat{R} - \frac{1}{2}(\partial\hat{\psi})^2 - \frac{1}{12}e^{2\beta\hat{\psi}}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{G}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right), \quad (3.4)$$

where $\hat{G} = d\hat{B}$.

Its further truncation is clear from the field equations

$$\nabla^2 \hat{\psi} = \frac{\beta}{6}e^{2\beta\hat{\psi}}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{G}^{\hat{\mu}\hat{\nu}\hat{\rho}}, \quad (3.5)$$

$$D_{\hat{\rho}}(e^{2\beta\hat{\psi}}\hat{G}^{\hat{\mu}\hat{\nu}\hat{\rho}}) = 0, \quad D_{\hat{\rho}}\tilde{\hat{G}}^{\hat{\mu}\hat{\nu}\hat{\rho}} = 0. \quad (3.6)$$

These are consistent with setting the dilaton to zero and imposing a self-duality constraint on the three-form field, in which case the right hand side of Eq. (3.5) vanishes:

$$\hat{\psi} = 0, \quad \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \tilde{\hat{G}}_{\hat{\mu}\hat{\nu}\hat{\rho}}. \quad (3.7)$$

This gives the bosonic lagrangian of six-dimensional minimal supergravity :

$$\mathcal{L}_{MSG}^6 = \sqrt{\hat{g}} \left(\hat{R} - \frac{1}{12}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{G}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right), \quad (3.8)$$

with the three-form field

$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} \equiv 3\hat{B}_{[\hat{\mu}\hat{\nu},\hat{\lambda}]} \quad (3.9)$$

constrained by the self-duality condition

$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\lambda}} = \frac{1}{6}\sqrt{\hat{g}}\varepsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}\hat{\sigma}\hat{\tau}}\hat{G}^{\hat{\rho}\hat{\sigma}\hat{\tau}}. \quad (3.10)$$

Toroidal reduction of this theory to three dimensions leads to the coset $O(4,3)/(O(4) \times O(3))$, for which it is an anomalous oxidation point, as was pointed out in [33].

Finally this theory can be truncated to the six-dimensional vacuum Einstein theory, which is an ultimate oxidation point of the coset for the group $SL(4,R) \sim O(3,3)$. To summarize, the six-dimensional truncation sequence is

$$EMDA \longrightarrow EMD \longrightarrow MSG \longrightarrow VG. \quad (3.11)$$

IV. REDUCTION TO $D = 5$

The above six-dimensional theories being compactified on S^1 give rise to five-dimensional theories with the same hidden symmetries. The parent theory $O(5,4)/(O(5) \times O(4))$ leads to a five-dimensional theory containing three vectors, three scalars and a two-form field. This is beyond our list, but can be further truncated to other $D = 5$ theories according to the scheme

$$\begin{array}{ccc}
 O(5,4) & & \{3A_\mu, 3S, B_{\mu\nu}\} \\
 \swarrow & \searrow & \swarrow \quad \searrow \\
 O(4,4) & O(5,3) & U(1)^3 SG \quad EM_2DA \\
 \searrow & \swarrow & \searrow \quad \swarrow \\
 & O(4,3) & EMDA \\
 \swarrow & \searrow & \swarrow \quad \searrow \\
 G_{2(2)} & O(3,3) & MSG \quad EDA \\
 \searrow & \swarrow & \searrow \quad \swarrow \\
 & SL(3,R) & VG
 \end{array} \quad (4.1)$$

Apart from the cosets listed in the previous section this includes

- $O(5,3)/(O(5) \times O(3))$ (dimension 15),
- $G_{2(2)}/(SU(2) \times SU(2))$ (dimension 8),

- $SL(3, R)/(O(3) \times O(3))$ (dimension 5).

Actually, the right subchain via the EM_2DA sequence simply corresponds to successive dropping of vector fields and we do not give the details here. The left subchain via $U(1)^3 SG$ is rather non-trivial and can be alternatively regarded as compactification of the truncated $D = 6$ theories with $G = O(4, 4)$ (EDA) and $G = O(4, 3)$ (MSG).

A. Reducing $D = 6$ EDA to $D = 5$ $U(1)^3$ supergravity

Starting with the EDA model in six dimensions (3.4) and reducing relative to a Killing vector ∂_z ,

$$ds_6^2 = e^{2\alpha\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{-6\alpha\phi} (dz + \mathcal{A}_\mu dx^\mu)^2, \quad (4.2)$$

$$\hat{B} = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu + A_\nu dx^\nu \wedge dz \quad (4.3)$$

where $\alpha^2 = 1/24$ and μ, ν are five-dimensional indices, we obtain

$$\begin{aligned} \mathcal{L}_5 = & \sqrt{g_5} \left(R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} (\partial\psi)^2 - \frac{1}{4} e^{-8\alpha\phi} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \right. \\ & \left. - \frac{1}{4} e^{2\beta\psi+4\alpha\phi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} e^{2\beta\psi-4\alpha\phi} G_{\mu\nu\rho} G^{\mu\nu\rho} \right), \end{aligned} \quad (4.4)$$

where $F = dA$, $G = dB - F \wedge \mathcal{A}$. Variation over B gives

$$D_\mu \left(e^{2\beta\psi-4\alpha\phi} G^{\mu\nu\rho} \right) = 0, \quad (4.5)$$

which is solved by

$$G^{\mu\nu\rho} = e^{-2\beta\psi+4\alpha\phi} \frac{1}{2\sqrt{g_5}} \epsilon^{\mu\nu\rho\sigma\tau} K_{\sigma\tau} = e^{-2\beta\psi+4\alpha\phi} \tilde{K}^{\mu\nu\rho}, \quad (4.6)$$

with $K = dC$. Other equations with account for (4.5, 4.6) read

$$D_\mu \left(e^{2\beta\psi+4\alpha\phi} F^{\mu\nu} \right) = -\frac{1}{2} \tilde{K}^{\nu\mu\rho} \mathcal{F}_{\mu\rho}, \quad (4.7)$$

$$D_\mu \left(e^{-8\alpha\phi} \mathcal{F}^{\mu\nu} \right) = -\frac{1}{2} \tilde{K}^{\nu\mu\rho} F_{\mu\rho}. \quad (4.8)$$

The Bianchi identity for G can be rewritten similarly:

$$D_\mu \left(e^{-2\beta\psi-4\alpha\phi} K^{\mu\nu} \right) = -\frac{1}{2} \tilde{F}^{\nu\mu\rho} \mathcal{F}_{\mu\rho}. \quad (4.9)$$

The scalar equations are

$$\nabla^2 \phi = -2\alpha e^{-8\alpha\phi} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \alpha e^{2\beta\psi+4\alpha\phi} F_{\mu\nu} F^{\mu\nu} - \frac{\alpha}{3} e^{2\beta\psi-4\alpha\phi} G_{\mu\nu\rho} G^{\mu\nu\rho}, \quad (4.10)$$

$$\nabla^2 \psi = \frac{\beta}{2} e^{2\beta\psi+4\alpha\phi} F_{\mu\nu} F^{\mu\nu} + \frac{\beta}{6} e^{2\beta\psi-4\alpha\phi} G_{\mu\nu\rho} G^{\mu\nu\rho}, \quad (4.11)$$

and the Einstein equations read

$$R_{\mu\nu} = \frac{1}{2} \left\{ \partial_\mu \phi \partial_\nu \phi + \partial_\mu \psi \partial_\nu \psi + e^{-8\alpha\phi} \mathcal{F}_{\mu\lambda} \mathcal{F}_\nu{}^\lambda + e^{2\beta\psi+4\alpha\phi} F_{\mu\lambda} F_\nu{}^\lambda + \frac{1}{2} e^{2\beta\psi-4\alpha\phi} G_{\mu\lambda\rho} G_\nu{}^{\lambda\rho} \right\}. \quad (4.12)$$

Define three scalars X_I , $I = 1, 2, 3$, constrained by $X_1 X_2 X_3 = 1$ as

$$\ln X_1 = \beta\psi - 2\alpha\phi, \quad \ln X_2 = -\beta\psi - 2\alpha\phi, \quad \ln X_3 = 4\alpha\phi, \quad (4.13)$$

and denote $F_{\mu\nu}^I = (K_{\mu\nu}, F_{\mu\nu}, \mathcal{F}_{\mu\nu})$ with potentials $F^I = dA^I$. Then the above system of equations can be derived from the action of $U(1)^3 D = 5$ supergravity:

$$\mathcal{L}_{U3SG}^5 = \sqrt{g_5} \left(R - \frac{1}{2} \mathcal{G}_{IJ} (\partial X^I) (\partial X^J) - \frac{1}{4} \mathcal{G}_{IJ} F_{\mu\nu}^I F_{\mu\nu}^J \right) - \frac{1}{24} \epsilon^{\mu\nu\lambda\rho\tau} \delta_{IJK} F_{\mu\nu}^I F_{\lambda\rho}^J A_\tau^K, \quad (4.14)$$

where $\mathcal{G}_{IJ} = \text{diag}((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2})$, $\delta_{IJK} = 1$ for the permutation of the indices 1, 2, 3 and zero otherwise.

B. Reducing self-dual $D = 6$ supergravity

Now we start with the six-dimensional theory (3.8). The six-dimensional Einstein equations read:

$$\hat{R}_{\hat{\nu}}{}^{\hat{\mu}} - \frac{1}{2} \delta_{\hat{\nu}}{}^{\hat{\mu}} \hat{R} = \frac{1}{4} \hat{G}^{\hat{\mu}\hat{\rho}\hat{\sigma}} \hat{G}_{\hat{\nu}\hat{\rho}\hat{\sigma}}. \quad (4.15)$$

Note that $\hat{G}^{\hat{\tau}\hat{\rho}\hat{\sigma}} \hat{G}_{\hat{\tau}\hat{\rho}\hat{\sigma}} = 0$ for a self-dual 3-form².

To reduce to five dimensions we use the same ansatz (4.2) for the metric and slightly modify the decomposition of the two-form potential

$$\hat{B} = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{\sqrt{2}} A_\nu dz \wedge dx^\nu \quad (4.16)$$

leading to

$$\hat{G}_{\mu\nu\lambda} = 3B_{[\mu\nu,\lambda]}, \quad \hat{G}_{\mu\nu 6} = \sqrt{2} A_{[\mu,\nu]} \equiv -\frac{1}{\sqrt{2}} F_{\mu\nu}. \quad (4.17)$$

Self-duality of \hat{G} gives

$$\hat{G}^{\mu\nu\lambda} = \frac{1}{2\sqrt{2}\sqrt{g_5}} e^{-2\alpha\phi} \epsilon^{\mu\nu\lambda\rho\sigma} F_{\rho\sigma} \equiv e^{-2\alpha\phi} \frac{1}{\sqrt{2}} \tilde{F}^{\mu\nu\lambda}, \quad (4.18)$$

² The lagrangian (3.8) itself does not imply the self-duality condition, which should be imposed by hand after the variation is performed. This is enough for our purposes since actually we deal with the classical equations of motion. For a consistent Lorentz-covariant lagrangian formulation of theories involving chiral form fields see [38]. We thank Dmitri Sorokin for discussion about this point.

so the six-dimensional equation $\hat{G}_{;\lambda}^{\mu\nu\lambda} = 0$ is equivalent to the five-dimensional identity

$$D_\lambda \tilde{F}^{\mu\nu\lambda} \equiv 0. \quad (4.19)$$

Also,

$$-F_{\mu\nu} = -\frac{1}{2}\sqrt{g_5}e^{-4\alpha\phi}\varepsilon_{\mu\nu\rho\sigma\tau}(\sqrt{2}B^{[\rho\sigma,\tau]} + \mathcal{A}^\rho F^{\sigma\tau}). \quad (4.20)$$

leading to

$$D_\nu(e^{4\alpha\phi}F^{\mu\nu}) = \frac{1}{2}\mathcal{F}_{\nu\rho}\tilde{F}^{\mu\nu\rho} = \frac{1}{2}e^{8\alpha\phi}F_{\nu\rho}H^{\mu\nu\rho}, \quad (4.21)$$

with

$$\mathcal{F}_{\mu\nu} \equiv 2\mathcal{A}_{[\nu,\mu]}, \quad H^{\mu\nu\rho} \equiv -e^{-8\alpha\phi}\tilde{\mathcal{F}}^{\mu\nu\rho}. \quad (4.22)$$

These last definitions imply

$$D_\lambda(e^{8\alpha\phi}H^{\mu\nu\lambda}) = 0. \quad (4.23)$$

Now one uses the mixed components of the Einstein equations:

$$\begin{aligned} \hat{R}_6^\mu &\equiv \frac{1}{2}e^{-2\alpha\phi}D_\nu(e^{-8\alpha\phi}\mathcal{F}^{\mu\nu}) \\ &\equiv \frac{1}{12\sqrt{g_5}}e^{-2\alpha\phi}\partial_\nu(\varepsilon^{\mu\nu\rho\sigma\tau}H_{\rho\sigma\tau}) \\ &= -\frac{1}{8}e^{-2\alpha\phi}F_{\nu\rho}\tilde{F}^{\mu\nu\rho}, \end{aligned} \quad (4.24)$$

implying

$$H_{\mu\nu\rho} = 3(C_{[\mu\nu,\rho]} + \frac{1}{2}F_{[\mu\nu}A_{\rho]}). \quad (4.25)$$

The Einstein equation for R_{66} gives:

$$\nabla^2\phi = \frac{2\alpha}{3}e^{8\alpha\phi}H^{\tau\rho\sigma}H_{\tau\rho\sigma} + \alpha e^{4\alpha\phi}F^{\rho\sigma}F_{\rho\sigma}, \quad (4.26)$$

which is consistent with the equations of motion following from the following five-dimensional EMDA Lagrangian

$$\mathcal{L}_{EMDA}^5 = \sqrt{g_5}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{4\alpha\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{12}e^{8\alpha\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right). \quad (4.27)$$

V. TRUNCATIONS IN $D = 5$

Consistent truncations in five dimensions are governed by the double chain shown in (4.1). The right subchain is rather straightforward: to go from EM_2DA to $EMDA$ and then to EDA amounts to dropping the first and the second vectors, and further dropping of the two-form leads to VG . The left subchain $O(4,4) \longrightarrow O(4,3) \longrightarrow G_{2(2)} \longrightarrow SL(3,R)$ is not so trivial. In terms of theories this means

$$U(1)^3 SG \longrightarrow EMDA \longrightarrow MSG \longrightarrow VG. \quad (5.1)$$

Thus $D = 5$ $U(1)^3$ supergravity, whose $D = 3$ reduction is the coset $O(4,4)/O(4) \times O(4)$, can be consistently truncated to minimal $D = 5$ supergravity via an intermediate $D = 5$ $EMDA$ theory (an oxidation point of the coset $O(4,3)/(O(4) \times O(3))$). The direct truncation $U(1)^3 SG \longrightarrow MSG$ is in fact much simpler and amounts to the identifications

$$X_1 = X_2 = X_3 = 1, \quad A^1 = A^2 = A^3. \quad (5.2)$$

However, passing through the intermediate $EMDA$ step gives further dualities in our list of theories. So, first we make the identifications (following from (3.7))

$$X_1 = X_2 \equiv X, \quad X_3 = \frac{1}{X^2}, \quad A^1 = A^2 \equiv \frac{1}{\sqrt{2}}A, \quad (5.3)$$

arriving at the $EMDA$ theory (4.27). The field equations deriving from this Lagrangian are:

$$D_\rho(e^{8\alpha\phi}H^{\mu\nu\rho}) = 0, \quad (5.4)$$

$$D_\nu(e^{4\alpha\phi}F^{\mu\nu} + e^{8\alpha\phi}H^{\mu\nu\rho}A_\rho) = 0, \quad (5.5)$$

$$\nabla^2\phi = \alpha e^{4\alpha\phi}F_{\mu\nu}F^{\mu\nu} + \frac{2\alpha}{3}e^{8\alpha\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}, \quad (5.6)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2}\left\{\partial_\mu\phi\partial_\nu\phi + e^{4\alpha\phi}F_{\mu\lambda}F_\nu{}^\lambda + \frac{1}{2}e^{8\alpha\phi}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - g_{\mu\nu}\left[\frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi + \frac{1}{4}e^{4\alpha\phi}F_{\lambda\rho}F^{\lambda\rho} + \frac{1}{12}e^{8\alpha\phi}H_{\lambda\rho\sigma}H^{\lambda\rho\sigma}\right]\right\}, \quad (5.7)$$

together with the consistency conditions following from the definitions of the two- and three-forms,

$$D_\rho\tilde{F}^{\mu\nu\rho} = 0, \quad (5.8)$$

$$D_\nu(\tilde{H}^{\mu\nu} - \frac{1}{2}\tilde{F}^{\mu\nu\rho}A_\rho) = 0. \quad (5.9)$$

Equation (5.4) is solved by

$$H^{\mu\nu\rho} = \frac{1}{\sqrt{2}}e^{-8\alpha\phi}\tilde{K}^{\mu\nu\rho}, \quad (K_{[\sigma\tau,\lambda]} = 0). \quad (5.10)$$

Inserting this in (5.5), (5.9) and (5.6) gives

$$D_\nu \left(e^{4\alpha\phi} F^{\mu\nu} + \frac{1}{\sqrt{2}} \tilde{K}^{\mu\nu\rho} A_\rho \right) = 0, \quad (5.11)$$

$$D_\nu \left(e^{-8\alpha\phi} K^{\mu\nu} + \frac{1}{\sqrt{2}} \tilde{F}^{\mu\nu\rho} A_\rho \right) = 0, \quad (5.12)$$

$$\nabla^2 \phi = \alpha \left(e^{4\alpha\phi} F_{\mu\nu} F^{\mu\nu} - e^{-8\alpha\phi} K_{\mu\nu} K^{\mu\nu} \right), \quad (5.13)$$

which are consistent with the constraints

$$\phi = 0, \quad K_{\mu\nu} = \pm F_{\mu\nu}. \quad (5.14)$$

The remaining field equations

$$D_\nu \left(F^{\mu\nu} \pm \frac{1}{\sqrt{2}} \tilde{F}^{\mu\nu\rho} A_\rho \right) = 0, \quad (5.15)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{3}{4} \left(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \right) \quad (5.16)$$

can be transformed by the rescaling

$$F_{\mu\nu} = \pm \sqrt{\frac{2}{3}} f_{\mu\nu} \quad (5.17)$$

into the equations of minimal five-dimensional supergravity, deriving from the Lagrangian

$$\mathcal{L}_{MSG}^5 = \sqrt{g} \left(R - \frac{1}{4} f^{\mu\nu} f_{\mu\nu} \right) - \frac{1}{12\sqrt{3}} \varepsilon^{\mu\nu\rho\sigma\lambda} f_{\mu\nu} f_{\rho\sigma} a_\lambda. \quad (5.18)$$

The three-dimensional invariance group is $G_{2(2)}$. This can be further truncated to five-dimensional vacuum gravity, with three-dimensional group $SL(3, R)$, by setting the 2-form f to zero.

VI. TRUNCATIONS IN FOUR DIMENSIONS

The four-dimensional theories in the list (2.6) can be obtained by truncation of two compactified five-dimensional theories via several chains:

$$\begin{array}{ccc}
 O(5,3) & O(4,4) & \{4A_\mu, 4S, B_{\mu\nu}\} \quad \{4A_\mu, 6S\} \\
 \swarrow \quad \searrow \quad \swarrow & & \swarrow \quad \searrow \quad \swarrow \\
 O(5,2) & O(4,3) & EM_3DA \quad LLPT \\
 \searrow \quad \swarrow \quad \searrow & & \searrow \quad \swarrow \quad \downarrow \\
 SU(2,2) = O(4,2) & O(3,3) = SL(4, R) & EM_2DA \quad CGMS \\
 \downarrow & \searrow \quad \swarrow & \downarrow \quad \searrow \quad \downarrow \\
 SU(2,1) & O(3,2) & EM \quad EMDA \\
 \downarrow & \downarrow & \downarrow \quad \downarrow \\
 SL(2, R) & \leftarrow O(2,2) & VG \leftarrow EDA
 \end{array} \quad (6.1)$$

The first line indicates four-dimensional oxidation points of the cosets of $O(5,3)$ and $O(4,4)$, these are beyond the list (2.6). Lower lines correspond to different truncation schemes of these. In what follows we concentrate on the $O(4,3)$ subchain and do not give details of the subchain going through $O(5,2)$. There is also another chain leading to four-dimensional Einstein-Maxwell theory via $D = 5$ minimal supergravity, this will be discussed below separately.

A. $O(4,3)$ theory

Reduction of the $O(4,3)$ model from five to four dimensions leads to the Lagrangian obtained by Lavrinenko, Lu, Pope and Tran (LLPT) [39]

$$\begin{aligned} \mathcal{L}_{LLPT} = & \sqrt{g_4} \left(R - \frac{1}{2} \left[(\partial\phi)^2 + (\partial\varphi)^2 + e^{2\phi} (\partial\chi)^2 + e^{\sqrt{2}\phi} (\partial\sigma)^2 \right] \right. \\ & - \frac{1}{4} e^{-\phi} \left[e^{\sqrt{2}\phi} F_{\mu\nu}^+ F^{+\mu\nu} + F_{\mu\nu}^0 F^{0\mu\nu} + e^{-\sqrt{2}\phi} F_{\mu\nu}^- F^{-\mu\nu} \right] \\ & \left. - \frac{1}{2} \chi \left[\tilde{F}_{\mu\nu}^+ F^{-\mu\nu} + \frac{1}{2} \tilde{F}_{\mu\nu}^0 F^{0\mu\nu} \right] \right), \end{aligned} \quad (6.2)$$

with three Maxwell fields

$$\begin{aligned} F_{\mu\nu}^- &= 2A_{[\nu,\mu]}^-, \\ F_{\mu\nu}^0 &= 2 \left(A_{[\nu,\mu]}^0 + \sigma A_{[\nu,\mu]}^- \right), \\ F_{\mu\nu}^+ &= 2 \left(A_{[\nu,\mu]}^+ - \sigma A_{[\nu,\mu]}^0 - \frac{1}{2} \sigma^2 A_{[\nu,\mu]}^- \right). \end{aligned} \quad (6.3)$$

This theory can be truncated to $D = 4$ EM_2DA with two vector fields, which has the hidden symmetry $O(4,2) \sim SU(2,2)$.

The field equations for the dilaton ϕ and the axion σ are

$$\begin{aligned} \nabla^2 \phi = & \frac{1}{\sqrt{2}} \phi \left[e^{\sqrt{2}\phi} (\partial\sigma)^2 \right. \\ & \left. + \frac{1}{2} e^{-\phi} \left(e^{\sqrt{2}\phi} F_{\mu\nu}^+ F^{+\mu\nu} - e^{-\sqrt{2}\phi} F_{\mu\nu}^- F^{-\mu\nu} \right) \right], \end{aligned} \quad (6.4)$$

$$D_\mu \left[e^{\sqrt{2}\phi} \partial^\mu \sigma \right] = F_{\mu\nu}^0 \left[e^{\sqrt{2}\phi - \phi} F^{+\mu\nu} - e^{-\phi} F^{-\mu\nu} \right]. \quad (6.5)$$

These equations are consistent with the identifications

$$\phi = 0, \quad \sigma = 0, \quad F_{\mu\nu}^- = F_{\mu\nu}^+, \quad (6.6)$$

while the equations for the two-forms, written for $\varphi = \sigma = 0$,

$$D_\nu [e^{-\phi} F^{+\mu\nu} + \chi \tilde{F}^{-\mu\nu}] = 0, \quad (6.7)$$

$$D_\nu [e^{-\phi} F^{0\mu\nu} + \chi \tilde{F}^{0\mu\nu}] = 0, \quad (6.8)$$

$$D_\nu [e^{-\phi} F^{-\mu\nu} + \chi \tilde{F}^{+\mu\nu}] = 0 \quad (6.9)$$

are also consistent with the identification of the two-forms F^+ and F^- . The truncated Lagrangian

$$\begin{aligned} \mathcal{L}_4 = \sqrt{g_4} & \left(R - \frac{1}{2} [(\partial\phi)^2 + e^{2\phi}(\partial\chi)^2] \right. \\ & \left. - \frac{1}{4} [e^{-\phi} F_{\mu\nu}^a F^{a\mu\nu} + \chi \tilde{F}_{\mu\nu}^a F^{a\mu\nu}] \right) \end{aligned} \quad (6.10)$$

(with $F^1 \equiv F^0$ and $F^2 \equiv \sqrt{2}F^\pm$ is that of the EM_2DA model [15, 16], which after reduction to three dimensions has the symmetry $O(4,2) \sim SU(2,2)$. EM_2DA can then be truncated to $p = 1$ $EMDA$ (hidden symmetry group $O(3,2) \sim Sp(4,R)$) by identifying the two 2-forms (or setting one of them to zero) [16]. In turn, $EMDA$ can be truncated to four-dimensional EDA , with three-dimensional group $O(2,2) \sim SL(2,R) \times SL(2,R)$, by setting the remaining two-form to zero, and finally to four-dimensional vacuum gravity.

EM_2DA can also be truncated to four-dimensional Einstein-Maxwell theory, with three-dimensional invariance group $SU(2,1)$ by the consistent constraints

$$\phi = 0, \quad \chi = 0, \quad F_{\mu\nu}^2 = \tilde{F}_{\mu\nu}^1. \quad (6.11)$$

Finally, Einstein-Maxwell theory can be truncated to four-dimensional vacuum gravity (three-dimensional group $O(2,1) \sim SL(2,R)$).

B. CGMS subchain

Another chain of reduction/truncations shown in (6.1) goes from $LLPT$ to the theory obtained by Chen, Gal'tsov, Maeda and Sharakin [17] (CGMS) as a compactification of the five-dimensional EDA , which in turn is a compactification of six-dimensional vacuum gravity [18]). This amounts to setting

$$\varphi = \psi/\sqrt{2}, \quad \sigma = 0, \quad F_{\mu\nu}^+ = \mathcal{F}_{\mu\nu}, \quad F_{\mu\nu}^- = F_{\mu\nu}, \quad F_{\mu\nu}^0 = 0, \quad (6.12)$$

which is also consistent with the equations of motion (6.4). The resulting lagrangian is [17]:

$$\mathcal{L}_{CGMS} = \sqrt{g_4} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 - \frac{1}{4}(\partial\psi)^2 - \frac{1}{4}e^{-\phi} [e^\psi \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + e^{-\psi} F_{\mu\nu} F^{\mu\nu}] - \frac{\chi}{2} \tilde{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right). \quad (6.13)$$

Subsequent truncation to *EMDA* with one vector field amounts to setting

$$\psi = 0, \quad \mathcal{F}_{\mu\nu} = F_{\mu\nu}. \quad (6.14)$$

The remaining segment $EMDA \longrightarrow EDA \longrightarrow VG$ is the same as in the previous subsection. This subchain does not include Einstein-Maxwell theory.

C. From G_2 to EM

Meanwhile there is another reduction/truncation chain from five to four dimensions, not shown on the table (6.1), leading to $D = 4$ Einstein-Maxwell theory. Namely, it can also be obtained by truncation of minimal five-dimensional supergravity reduced to four dimensions since $SU(2, 1) \subset G_{2(2)}$: the corresponding sequence is

$$O(4, 3) \longrightarrow G_{2(2)} \longrightarrow SU(2, 1) \longrightarrow O(2, 1). \quad (6.15)$$

One starts with the five-dimensional action (5.18). Assuming the existence of a space-like Killing vector ∂_z , one can parametrize the five-dimensional metric and the Maxwell field A_5 by

$$ds_5^2 = e^{-2\phi} (dz + C_\mu dx^\mu)^2 + e^\phi ds_4^2, \quad (6.16)$$

$$A_5 = A_\mu dx^\mu + \sqrt{3}\kappa dz. \quad (6.17)$$

The corresponding four-dimensional lagrangian reads

$$\mathcal{L}_4 = \sqrt{g_4} \left[R - \frac{3}{2}(\partial\phi)^2 - \frac{3}{2}e^{2\phi}(\partial\kappa)^2 - \frac{1}{4}e^{-3\phi}G^2 - \frac{1}{4}e^{-\phi}\bar{F}^2 - \frac{1}{2}\kappa F\tilde{F} \right], \quad (6.18)$$

where

$$G = dC, \quad F = dA, \quad \bar{F} = F + \sqrt{3}C \wedge d\kappa, \quad (6.19)$$

and \tilde{F} is the four-dimensional Hodge dual of F . This describes an Einstein theory with two coupled abelian gauge fields F and G , a dilaton ϕ and an axion κ . The field equations in terms of the four-dimensional variables read

$$\nabla^2\phi - e^{2\phi}(\partial\kappa)^2 + \frac{1}{4}e^{-3\phi}G^2 + \frac{1}{12}e^{-\phi}\bar{F}^2 = 0, \quad (6.20)$$

$$\nabla_\mu (e^{2\phi}\nabla^\mu\kappa) - \frac{1}{3} \left[\sqrt{3}\nabla_\mu (e^{-\phi}\bar{F}^{\mu\nu}C_\nu) + \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu} \right] = 0, \quad (6.21)$$

$$\nabla_\mu (e^{-\phi}\bar{F}^{\mu\nu} + 2\kappa\tilde{F}^{\mu\nu}) = 0, \quad (6.22)$$

$$\nabla_\mu (e^{-3\phi}G^{\mu\nu}) + \sqrt{3}e^{-\phi}\bar{F}^{\mu\nu}\partial_\mu\kappa = 0. \quad (6.23)$$

The consistent truncation to the Einstein-Maxwell theory is achieved via the constraints

$$\phi = 0, \quad \kappa = 0, \quad G_{\mu\nu} = \frac{1}{2\sqrt{3}} \sqrt{g_4} \varepsilon_{\mu\nu\rho\sigma} F_{(4)}^{\rho\sigma}. \quad (6.24)$$

Then the first two equations are trivially satisfied, while the last two become the Maxwell equation and Bianchi identity for $F_{\mu\nu}$, the Lagrangian reducing to that of the *EM* theory.

In a recent paper [40] we have derived a new simple matrix representative of the coset $O(4,3)/O(4) \times O(3)$, which can also be used in the case of minimal five-dimensional supergravity. The truncations (6.15) are implemented there, imposing algebraic constraints on the potentials.

It is worth noting that the $D = 3$ *EM* coset $SU(2,1)/S(U(2) \times U(1))$ has $D = 4$ as an oxidation endpoint [33], so that this theory cannot be obtained by toroidal dimensional reduction from higher-dimensional theories. However, as we have shown, it can be embedded into higher-dimensional theories using reduction/truncation chains.

VII. CONCLUSIONS

We have described various mappings between six, five and four-dimensional theories whose $D = 3$ cosets are subspaces of $O(5,4)/(O(5) \times O(4))$. Among them are truncated heterotic effective actions (*EMDA*), minimal supergravities in five and six dimensions, $U(1)^3$ five-dimensional supergravity and some popular four-dimensional models. These mappings open a way to construct exact solutions in five and (especially) in six dimensions starting from known four-dimensional solutions. For this purpose we have given a detailed description of reduction/truncation chains in term of natural non-reduced field variables. Using them one can easily uplift lower-dimensional solutions to higher dimensions. Combining this with coset transformations and generation of solutions as geodesics of coset spaces, one acquires new tools for solution generation. Applications will be given elsewhere.

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